Lesson 9 - Partial Fractions, Part I

I $\int \frac{1}{ax+b}\,dx$ and $\int \frac{1}{(ax+b)^n}\,dx$

II. Motivation

III. Partial Fractions Decomposition

IV. Examples

I. $\int \frac{1}{ax+b}\,dx \qquad \int \frac{1}{(ax+b)^n}\,dx$

a, b constants

$a \neq 0$

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(|ax+b|) + C}$$

$$\boxed{\int \frac{1}{(ax+b)^n}\,dx = \int (ax+b)^{-n}\,dx = \frac{1}{a}\frac{(ax+b)^{-n+1}}{-n+1} + C \qquad n \neq 1}$$

Ex $\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln(|2x+3|) + C$

$a = 2 \quad b = 3$

$\int \frac{5}{4-7x}\,dx = 5\int \frac{1}{4-7x}\,dx = 5 \cdot \frac{-1}{7}\ln(|4-7x|) + C$

$= \frac{-5}{7}\ln(|4-7x|) + C$

II Motivation

$$\int \frac{x+7}{x^2+2x-3}\,dx$$

Toolbox: u-subs? What if

$$u = x^2+2x-3$$
$$du = (2x+2)\,dx$$
$$\frac{1}{2}du = \underbrace{(x+1)\,dx}_{\text{not in my integral}}$$

u-sub is not going to work

What if I tell you that

$$\frac{x+7}{x^2+2x-3} = \frac{2}{x-1} - \frac{1}{x+3}$$

So $\displaystyle\int \frac{x+7}{x^2+2x-3}\,dx = \int\left[\frac{2}{x-1} - \frac{1}{x+3}\right]dx$

$$= 2\ln(|x-1|) - \ln(|x+3|) + C$$

Partial Fractions Handout

III Partial Fractions Decomposition

MA 16020 LESSONS 9+10: PARTIAL FRACTIONS

METHOD OF DECOMPOSING INTO PARTIAL FRACTIONS

Given a rational function $\frac{N(x)}{D(x)}$

1. Factor the denominator as much as possible.

2. Write the fraction into decomposition form.

"terms" = terms in the factored denominator

   a) Distinct linear terms like $x - a$ decompose to
   $$\frac{A}{x-a}$$
   b) Repeated linear terms like $(x-a)^3$ decompose to
   $$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$
   c) Distinct irreducible quadratic terms like $x^2 + a^2$ decompose to
   $$\frac{Ax+B}{x^2+a^2}$$
   d) Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to
   $$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

3. Combine your decomposition from (2) as 1 fraction.

4. Set the original numerator, $N(x)$, equal to the numerator from (3).

5. Equate the coefficients of the terms, to yield a system of equations. Then solve the constants.
   i.e. Find $A, B, C, \ldots$

6. Plug the values found in (5) in (2).

$\boxed{\text{Ex}}$ Find the PFD (Partial Fractions Decomposition)
(Don't integrate)

a) $$\frac{2x-13}{x^2-x-2} = \frac{2x-13}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

$$\frac{2x-13}{x^2-x-2} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$(x-2)(x+1)$

So $2x-13 = A(x+1)+B(x-2)$

$2x-13 = Ax + A + Bx - 2B$

$2x-13 = (A+B)x + (A-2B)$

$$\begin{cases} A+B=2 & (1) \\ A-2B=-13 & (2) \end{cases}$$

(1) − (2)

$A + B = 2$

$-A + 2B = 13$

$3B = 15$

$B = 5$

$\downarrow$ (1)

$A + 5 = 2$

$A = -3$

$$\frac{2x-13}{x^2-x-2} = \frac{-3}{x-2} + \frac{5}{x+1}$$

b) $$\underbrace{\frac{2x^2+9x+9}{(x+1)^2(x+2)}}_{\text{C.D.}} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$$

$$= \frac{A(x+1)(x+2) + B(x+2) + C(x+1)^2}{(x+1)^2(x+2)}$$

$$= \frac{A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)}{(x+1)^2(x+2)}$$

$$= \frac{(A+C)x^2 + (3A+B+2C)x + (2A+2B+C)}{(x+1)^2(x+2)}$$

$$\begin{cases} A + C = 2 & (1) \\ 3A + B + 2C = 9 & (2) \\ 2A + 2B + C = 9 & (3) \end{cases}$$

1st

$-2(2) + (3)$

$-6A - \cancel{2B} - 4C = -18$

$2A + \cancel{2B} + C = 9$

$-4A - 3C = -9 \quad (4)$

2nd

(1) $A + C = 2$

(4) $-4A - 3C = -9$

$4(1) + (4)$

$\cancel{4A} + 4C = 8$

$-\cancel{4A} - 3C = -9$

$\boxed{C = -1}$

3rd

$$\boxed{C=-1 \rightarrow ① \quad A+(-1)=2 \quad \boxed{A=3}}$$

$C=-1,\ A=3 \rightarrow ②$

$3(3)+B+2(-1)=9$

$B-2=0 \Rightarrow \boxed{B=2}$

So $\dfrac{2x^2+9x+9}{(x+1)^2(x+2)} = \dfrac{3}{(x+1)} + \dfrac{2}{(x+1)^2} - \dfrac{1}{(x+2)}$

$$\Rightarrow \int \frac{2x^2+9x+9}{(x+1)^2(x+2)}\,dx = \int\left[\frac{3}{(x+1)} + \underbrace{\frac{2}{(x+1)^2}}_{2(x+1)^{-2}} - \frac{1}{(x+2)}\right]dx$$

$$= 3\ln(|x+1|) + 2\,\frac{(x+1)^{-1}}{-1} - \ln(|x+2|) + C$$

$$= 3\ln(|x+1|) - \frac{2}{(x+1)} - \ln(|x+2|) + C$$